Published by Institute of Physics Publishing for SISSA/ISAS
Received: *January 30, 2004*
Revised: *March 4, 2004*
Accepted: *March 4, 2004*

Correspondence principle for black holes in plane waves

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Abstract: We compare the entropy as a function of energy of excited strings and black strings in an asymptotically plane wave background at the level of the correspondence principle. For the plane wave supported by the NSNS 3-form flux, neither the entropy formula nor the cross-over scale is affected by the presence of the flux and the correspondence is found to hold. For the plane wave supported by the RR 3-form flux, both the entropy and the cross-over point are modified, but the correspondence is still found to hold.

Keywords: Black Holes in String Theory, Penrose limit and pp-wave background.

Contents

1. Introduction

Plane waves have received increased attention recently as an important background space-time in string theory. These backgrounds are interesting for two reasons. On one hand, the theory on the world sheet admits a simple realization, making many explicit computations possible [1] even in the presence of a Ramond-Ramond background flux [2]. On the other hand, certain plane wave backgrounds [3] admit dual interpretation as a scaling limit of certain field theories [4]. This provides an exciting opportunity to explore gravitational physics, in an asymptotically plane wave background geometry, strictly in the framework of quantum field theory.

The most immediate application of this duality in addressing problems of gravity that comes to one's mind is the physics of Schwarzschild black holes. Consideration of Schwarzschild black holes in an asymptotically anti de-Sitter space-time played a significant role in clarifying the nature of holography in the context of AdS/CFT correspondence [5], and it is natural to expect that Schwarzschild black holes in an asymptotically plane wave space-time would equally clarify the nature of the correspondence of [4]. The first step in such a line of investigation is to construct an explicit black hole solution. Immediately following the proposal of [4], there have been numerous attempts to construct solutions of this type, with precisely this goal in mind.

Constructing Schwarzschild black hole solutions in an asymptotically plane wave space-time has proven to be extremely challenging. The difficulty stems from the fact that the symmetries of a background with a horizon and an asymptotically null background fluxes are not transparent. A review of the structure of horizons and plane waves can be found in [6].

The first explicit solution of Schwarzschild black strings in an asymptotically plane wave solution was written down very recently in a series of papers [7, 8]. (An explicit solution of a BPS black string in an asymptotically plane wave geometry was identified earlier in an inspiring paper [9].) Unfortunately, these black strings are embedded in a wrong asymptotic plane wave for the correspondence of [4] to be applicable. This correspondence relates a limit of $\mathcal{N}=4$ SYM to type-IIB string theory on a plane wave supported by the Ramond-Ramond *five-form flux* [3]. The solutions constructed in [7, 8] are that of black strings embedded in a plane wave supported either by the NSNS or the RR *three-form fluxes* [10, 11]. We are therefore unable to explore the properties of these black holes by studying the $\mathcal{N}=4$ SYM and using the correspondence of [4].

Despite this shortcoming, plane waves supported by three-form fluxes are still special in that the world sheet theory is simple. Of course, the application of full interacting string theory as a means to study the microscopic properties of Schwarzschild black holes is an important outstanding challenge, even if the world sheet theory is extremely simple (such as in the case of Minkowski space). There does exist, however, a scheme to semi-quantitatively compare the properties of *free* strings and black holes, sometimes referred to as the *correspondence principle* [12, 13]. The goal of this article is to explore the black string solutions of [7, 8] in this context.

Correspondence principle in plane wave space-time was also considered in [14], but its content appears to have very little overlap with what is presented in this paper.

The organization of this paper is as follows. We begin in section 2 by reviewing the construction of black string solutions [7, 8]. In section 3, we review the correspondence principle and how it applies to black strings in plane waves supported by the NSNS 3-form field strength. In section 4, we describe the string theory computation relevant for the application of the correspondence principle. In section 5, we examine the status of the correspondence principle for the plane waves supported by the RR 3-form field strengths. We will conclude in section 6.

It should be emphasized that the construction of a black hole/black string solution in an asymptotically plane wave space-time supported by the RR five-form remains an important outstanding problem. We hope that this problem will be solved in due course and will pave a path toward a quantitative study of black hole physics in terms of the $\mathcal{N}=4$ SYM theory.

2. Black strings in asymptotically plane wave space-time

In this section, we will review the construction of black string solutions first reported in [7, 8]. The main ingredient behind this construction is a set of manipulations which was originally formulated in [15] which we refer to as *Null Melvin Twist* following [8]. The reader should refer to [8] for the details of this manipulation. Here, we summarize the steps using a slightly different notation.

1. Consider a ten dimensional Minkowski space-time for a type-IIB supergravity theory, written in terms of coordinates t, y, ρ_i, ϕ_i:

$$ds^2 = -dt^2 + dy^2 + \sum_{i=1}^{4}(d\rho^2 + \rho^2 d\phi_i^2)\,. \tag{2.1}$$

2. Boost to a new frame

$$\begin{pmatrix} t \\ y \end{pmatrix} = \begin{pmatrix} \cosh\gamma & -\sinh\gamma \\ -\sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} t' \\ y' \end{pmatrix}. \tag{2.2}$$

3. Compactify y' so that it has radius R and T-dualize along y' so that the new coordinate $\tilde{y}'$ has radius α'/R.

4. "Twist," by replacing $d\phi_i$ by $d\phi_i + \omega_i\, d\tilde{y}'$ in the line elements.

5. T-dualize from $\tilde{y}'$ to y'.

6. Boost back to the original frame

$$\begin{pmatrix} t' \\ y' \end{pmatrix} = \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} t \\ y \end{pmatrix}. \tag{2.3}$$

At this stage, one arrives at a seemingly complicated space-time depending on parameters R, γ and ω_i for $i = 1\ldots 4$. All of these space-times, however, are related to Minkowski space by dualities, boosts, and twists, and belong to the class of exactly solvable string theories considered [16, 17]. For example, setting $\gamma = 0$ gives rise to the general NSNS Melvin solution. If instead we set all $\omega_i = \omega$ and

7. Scale γ to infinity, keeping

$$\eta = \frac{1}{2}\omega e^{\gamma} = \text{fixed}\,. \tag{2.4}$$

Then, the end result is a plane wave geometry

$$\begin{aligned} ds^2 &= -dt^2 + dy^2 - \eta^2 r^2 (dt+dy)^2 + \sum_{i=1}^{4}(d\rho_i^2 + \rho_i^2 d\phi_i^2) \\ e^{\varphi} &= 1 \\ B &= \eta(dt+dy)\wedge(\sum_{i=1}^{4}\rho_i^2 d\phi_i) \end{aligned} \tag{2.5}$$

where $r^2 = \sum_i \rho_i^2$.

Although we have so far only described Null Melvin Twists applied to Minkowski space-time, these manipulations can easily be applied to any space-time with a translational isometry for T-dualizing and a rotational isometry for twisting. One can, for example, apply the Null Melvin Twist to the Schwarzschild black string solution

$$ds^2 = -f(r)dt^2 + dy^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_7^2\,. \tag{2.6}$$

$$f(r) = 1 - \frac{M}{r^6}\,. \tag{2.7}$$

To apply the Null Melvin Twist, it may be more convenient to rewrite

$$\frac{1}{f(r)}dr^2 + r^2 d\Omega_7^2 \qquad \text{as} \qquad \frac{1-f(r)}{f(r)}dr^2 + \sum_i (d\rho_i^2 + \rho_i^2 d\phi_i^2)\,. \tag{2.8}$$

In the end, one obtains the space-time

$$\begin{aligned} ds^2_{str} &= -\frac{f(r)\,\left(1+\eta^2 r^2\right)}{k(r)}\,dt^2 - \frac{2\,\eta^2\,r^2\,f(r)}{k(r)}\,dt\,dy + \left(1 - \frac{\eta^2\,r^2}{k(r)}\right)\,dy^2 + \\ &\quad + \frac{dr^2}{f(r)} + r^2\,d\Omega_7^2 - \frac{\eta^2\,r^4\,(1-f(r))}{4\,k(r)}\,\sigma^2 \\ e^{\varphi} &= \frac{1}{\sqrt{k(r)}} \\ B &= \frac{\eta\,r^2}{2k(r)}\,\left(f(r)\,dt + dy\right)\wedge\sigma \end{aligned} \tag{2.9}$$

where

$$k(r) = 1 + \frac{\eta^2 M}{r^4}\,, \qquad \text{and} \qquad r^2\sigma = \frac{1}{2}\sum_{i=1}^{4}\rho_i^2 d\phi_i\,. \tag{2.10}$$

This is the supergravity solution for a Schwarzschild black string in an asymptotically plane wave space-time supported by an NSNS three form flux, whose magnitude is parameterized by η. A similar solution supported by the RR three from flux can be constructed immediately by S-dualizing the supergravity solution. These are the solutions that we will be considering in this paper.

Let us comment about the physical properties of these black strings. Among the most basic physical characteristics of black strings are its horizon area and its surface gravity. The horizon for the space-time (2.9) is located at $r_H^6 = M$. Its area in Einstein frame is readily computed to be

$$\mathcal{A} = 2\pi R\,M^{7/6}\Omega_7\,, \tag{2.11}$$

where we have assumed that the y coordinate is compactified[1] on a circle of radius R, and Ω_7 is the area of the unit 7-sphere. This area is independent of the background field strength parameterized by η.

Computing the surface gravity is a bit more subtle. It is defined as

$$\kappa^2 = -\frac{1}{2}(\nabla^a\xi^b)(\nabla_a\xi_b) \tag{2.12}$$

in Einstein frame, where ξ^a is the time-like Killing vector normal to the horizon (See e.g. [19]). In the case of solution (2.9), the vector

$$\xi^a = \left(\frac{\partial}{\partial t}\right)^a \tag{2.13}$$

[1]Such a compactification gives rise to closed time-like curves and may lead to additional subtleties of the type considered in [18]. We will not be addressing this point in this paper.

turns out to be the one normal to the horizon. For this choice of the time-like Killing vector, the surface gravity

$$\kappa^2 = \frac{36}{M^{1/3}} \tag{2.14}$$

also comes out to be independent of the parameter η. Of course, the precise value of the surface gravity and temperature depends on the normalization of the Killing vector ξ^a. We will *choose* to normalize ξ^a precisely as (2.13) in the coordinate where the metric asymptote to a geometry of the form given in (2.9). This means that in order to make a sensible comparison between black hole thermodynamics and the statistical mechanics of a microscopic system, one must evaluate the Boltzmann trace of the specific hamiltonian operator conjugate to the Killing vector (2.13).

3. Review of the correspondence principle

The correspondence principle states that in a weakly coupled string theory, an object of mass m has the physical properties of a classical black hole to a good approximation as long as the curvature of the metric in string frame near the horizon is smaller than the scale set by the string scale, and has the physical properties of an excited string if the Schwarzschild radius is smaller [12]. As a consequence of this principle, at the critical mass m where the Schwarzschild radius is of the order of the string scale, the entropy of the excited string in a flat background and the entropy of the black hole must also be of same order of magnitude [13].

This principle can easily be verified for black holes and black strings in asymptotically flat space-time. Let us consider, for the sake of concreteness, a neutral black string in ten space-time dimensions (2.6) wrapping a compact circle of radius R, which is equivalent to a black hole in nine space-time dimensions. The square of the Riemann curvature tensor $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$ scales as $1/r_H^4$, indicating that the expected cross-over scale is when r_H is of the order of l_s.[2]

The entropy of such a black hole is easily computed to be

$$S_{BH} = \frac{1}{G_9}(G_9 E)^{7/6} \tag{3.1}$$

where E is the energy associated with the black string

$$E = \frac{M}{G_9} \tag{3.2}$$

and is derived using the standard relation $dE = TdS$. G_9 is the nine dimensional Newton constant

$$G_9 = \frac{G_{10}}{R} = \frac{g_s^2 l_s^8}{R}\,. \tag{3.3}$$

[2]To be more precise, the square of the Riemann tensor does have a mild dependence on η, behaving like $f(\eta)/r_H^4$ where $f(\eta)$ is a function which smoothly interpolates between numbers of order one as η is varied from zero to infinity.

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The entropy of a string is dictated by the Hagedorn density of states

$$S_s = l_s E\,. \tag{3.4}$$

We have ignored the numerical factor of order one in this estimate as they are irrelevant in the context of the correspondence principle. The Schwarzschild radius is

$$r_H^6 = G_9 E\,. \tag{3.5}$$

One can immediately verify that the entropies

$$S_{BH} = S_s = \frac{l_s^7}{G_9} \tag{3.6}$$

match when

$$l_s^6 = r_H^6 = G_9 E\,. \tag{3.7}$$

This relation holds for any value of g_s provided that its value is of order one or less. This is the essence of the statement of the correspondence principle for black strings in an asymptotically flat space-time.

One additional technical comment is in order regarding the Gregory-Laflamme instability of the black string solution. Black strings are unstable to decay when the Schwarzschild radius r_H is of the order of the compactification radius R. This means that the black string is unstable at energies below $E = R^6/G_9$. In order for this instability to be hidden below the cross-over scale, R must be of the order or smaller than l_s. Of course, if R is smaller than l_s, one must worry about the Gregory-Laflamme instability of the T-dual picture. We are therefore forced to consider the case where the compactification radius R is of the order of the string scale l_s in order to apply the correspondence principle to the black string solution.

In the case of the black strings in an asymptotically plane wave background (2.9), we found in the previous section that the area of the horizon and the surface gravity are unaffected by the three form field strength parameter η. This implies that the entropy and the temperature are also unaffected by η. In order for the critical mass at the cross-over point to correspond to a black string whose Schwarzschild radius is the string scale, the entropy of the strings in the plane wave background should also be unaffected by η. This is what we will examine in the following section.

4. Thermal partition functions for strings in plane wave geometry

In this section we will describe the computation of the thermal partition function of strings in asymptotically plane wave background from which one can derive the formula for the entropy which enters in the consideration of the correspondence principle. Our goal is to compute the Boltzmann trace over for the spectrum of generator (2.13) for the set of string states in these backgrounds. Our task is drastically simplified in light of the fact that quantization of the strings and the computation of closely related one loop partition functions have been done already by many people [1, 16, 17],[20]–[32]. For the purpose of illustration, let us work with the bosonic theory and follow the notation of [16]. This paper considered string theory in a background defined by the sigma-model of the form

$$I = \frac{1}{\pi\alpha'\tau_2}\int d^2\sigma\left[F^{-1}(x)C\bar{C} + \bar{C}(\partial u' + A_1) - C(\bar{\partial}v' + A_2) + \sum_i \partial x_i'\bar{\partial}x_i'^*\right] \tag{4.1}$$

where

$$\begin{aligned} A_1 &= \partial y_* - \frac{i}{2}\sum_i \alpha_i(x_i'\partial x_i'^* - x_i'^*\partial x_i') \\ A_2 &= \bar{\partial}y_* + \frac{i}{2}\sum_i \beta_i(x_i'\bar{\partial}x_i'^* - x_i'^*\bar{\partial}x_i') \\ F^{-1}(x) &= 1 + \sum_i \alpha_i\beta_i x_i' x_i'^* \\ x_i' &= e^{i(q_{+i}y + q_{-i}t)}x_i\,. \end{aligned} \tag{4.2}$$

We have slightly generalized the model of [16] to allow non-vanishing values of α_i, β_i, q_{+i}, and q_{-i} for each of the 12 transverse planes of the bosonic string theory. Setting

$$\alpha_i = 2\eta\,, \qquad \beta_i = 0\,, \qquad q_{+i} = \eta\,, \qquad q_{-i} = -\eta\,. \tag{4.3}$$

gives rise to precisely the 26 dimensional version of the plane wave geometry (2.5).

The vacuum partition function in this background is

$$Z_0 = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\sum_{m,w}\int\frac{d\varepsilon}{2\pi}\,\mathrm{Tr}\left(e^{2\pi i(\tau L_0 - \bar{\tau}\bar{L}_0)}\right) \tag{4.4}$$

where

$$\varepsilon = -iE \tag{4.5}$$

is the imaginary extension of the zero mode along the time coordinate and sum over m and w corresponds to the momentum and the winding number along the compact y coordinates. The τ integral is done over the fundamental domain $\mathcal{F}$.

To compute the thermal partition function, on the other hand, we must evaluate

$$Z_T = \frac{1}{T}\int_{\mathcal{E}}\frac{d^2\tau}{\tau_2}\sum_{m,w}\int\frac{d\varepsilon}{2\pi}e^{ik'\varepsilon/T}\,\mathrm{Tr}\left(e^{2\pi i(\tau L_0 - \bar{\tau}\bar{L}_0)}\right)\Bigg|_{k'=-1} \tag{4.6}$$

where $\mathcal{E}$ refers to the semi-infinite strip $0 < \tau_1 < 1$, $0 < \tau_2 < \infty$. To see that this is the thermal partition function, write

$$2\pi i(\tau L_0 - \bar{\tau}\bar{L}_0) = -2\pi\tau_2(L_0 + \bar{L}_0) + 2\pi i\tau_1(L_0 - \bar{L}_0) \tag{4.7}$$

where the term proportional to τ_2 can be rewritten

$$L_0 + \bar{L}_0 = \frac{\alpha'}{2}(-E^2 + AE + B) \tag{4.8}$$

where A and B are simultaneously diagonalizable operator acting on the space of oscillating strings. Integrating out ε, τ_1, and τ_2 gives rise to

$$Z_T = \mathrm{Tr}\left(\exp\left(-\frac{1}{T}\left(\sqrt{B+\frac{A^2}{4}}+\frac{A}{2}\right)\right)\delta_{L_0-\bar{L}_0}\right) \tag{4.9}$$

which can readily be interpreted as the Boltzmann sum over the ensemble of excited single string states. This form of writing the thermal partition function can be related to the formulation in terms of light-cone hamiltonian as it appears for example in [25] by starting from (4.6) and following the sequence of steps

1. Decompactify the y coordinate by replacing m by $R\,p_y$, setting $w=0$, and sending R to zero.

2. Change variables by setting $p_y = p_+ - E$.

3. Set $E = i\varepsilon$ and integrate out ε and τ_2.

Decompactification however will give rise to a diverging volume factor. While overall normalization of the partition function is not particularly important for computing the density of states, we will make the point of compactifying y the coordinate on finite spatial circle, at least near $r=0$.

Details of the computation of the thermal partition function will be presented in the appendix. Here, let us simply quote the result that for the general model, the thermal partition function takes the form

$$\begin{aligned} Z_T = {} & \frac{RV_{22}}{(2\pi)^{23}\alpha'^{12}T}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\sum_{w,w'=\infty}^{\infty}\left(4\tau_2^{-1}\int d\lambda d\bar{\lambda}\right)\left(\frac{e^{4\pi\tau_2}}{\tau_2^{12}}|f(e^{2\pi i\tau})|^{-48}\right)\times \\ & \times\prod_{i=1}^{12}\left(\frac{1}{(2\pi)^2}\exp\left[-\frac{\pi(\chi_i-\tilde{\chi}_i)^2}{2\tau_2}\right]\frac{\tau_2|\theta_1'(0,\tau)|^2}{\theta_1(\chi_i|\tau)\theta_1(\tilde{\chi}_i|\bar{\tau})}\right)\times \\ & \times\exp\Bigg(-4\pi\tau_2^{-1}\Bigg(\lambda\bar{\lambda}-\left(\frac{1}{2}r(w'-\tau w)+\frac{i(k'-\tau k)}{4\pi\sqrt{\alpha'}T}\right)\bar{\lambda}+ \\ & \qquad +\left(\frac{1}{2}r(w'-\bar{\tau}w)-\frac{i(k'-\bar{\tau}k)}{4\pi\sqrt{\alpha'}T}\right)\lambda\Bigg)\Bigg) \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} \chi_i &= -\sqrt{\alpha'}[2\beta_i\lambda+q_{+i}r(w'-\tau w)]+\frac{iq_{-i}(k'-\tau k)}{2\pi T} \\ \tilde{\chi}_i &= -\sqrt{\alpha'}[2\alpha_i\bar{\lambda}+q_{+i}r(w'-\bar{\tau}w)]+\frac{iq_{-i}(k'-\bar{\tau}k)}{2\pi T}\,. \end{aligned} \tag{4.11}$$

The Hagedorn temperature associated with this thermal partition function can be extracted by studying the large τ_2 asymptotics of the integrand of the τ integral [33]. In this limit,

$$f(e^{2\pi i\tau})\to 1\,,\qquad \frac{\theta_1'(0,\tau)}{\theta_1(\chi_i,\tau)}\to\frac{\pi}{\sin(\pi\chi_i)}\,,\qquad \exp\left[-\frac{\pi(\chi_i-\tilde{\chi}_i)^2}{2\tau_2}\right]\to 1\,, \tag{4.12}$$

so that the λ integral becomes gaussian. After doing this integral, The Hagedorn temperature can be extracted by comparing the growth of $e^{4\pi\tau_2}$ and the decay of

$$\exp\left(-4\pi\tau_2^{-1}\frac{(\tau k)(\bar{\tau}k)}{16\pi^2\alpha' T^2}\right) \sim \exp\left(-\tau_2\frac{k^2}{4\pi\alpha' T^2}\right). \tag{4.13}$$

The critical temperature $T_{\rm crit}$ comes out to

$$T_{\rm crit} = \frac{k}{4\pi\sqrt{\alpha'}}, \tag{4.14}$$

which takes the smallest value

$$T_H = \frac{1}{4\pi\sqrt{\alpha'}} \tag{4.15}$$

for $k = 1$. This result is independent of η which only enters the partition function through the values of α_i, β_i, q_{+i}, and q_{-i}. So we find that the Hagedorn temperature for the plane wave is the same as the Hagedorn temperature of bosonic strings in Minkowski space. This was in fact pointed out first for the case of backgrounds compactified along a light-like direction in [27]. We therefore conclude that the entropy of strings in (2.5) is also given by (3.4). Since neither the black string entropy nor the string entropy were modified by the presence of η, the two will be of the same order of magnitude precisely when the Schwarzschild radius is of the order of the string length. In short, the correspondence principle is working.

5. Correspondence principle for black strings in a plane wave with Ramond-Ramond flux

So far, we have considered the cross-over in the physical properties of excited strings and a black string in the background of plane wave supported by the NSNS three form flux. Let us now consider the same issue for the case of plane wave background supported by the RR three form flux.

On the black string side, the supergravity solution corresponding to a black string in an asymptotically plane wave background with RR three form flux with strength μ can be constructed straightforwardly by S-dualizing the solution (2.9). As far as the physical properties such as the horizon area and the surface gravity is concerned, one should work with these space-time in the Einstein frame. The Einstein metric, however is invariant under the S-duality transformation, leading to the conclusion that the entropy and the temperature is independent of the strength of the background Ramond-Ramond three form μ.

On the perturbative string side, the generalization of (4.10) for type-IIB theory with RR-background can also be computed and takes the form

$$Z_T = \frac{R}{(2\pi)^2\alpha' T}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\sum_{\epsilon_i=0,1}\sum_{\substack{k\in 2\mathbb{Z}+\epsilon_1\\ k'\in 2\mathbb{Z}+\epsilon_2}}\sum_{w,w'\in\mathbb{Z}} e^{-\frac{1}{4\pi\alpha'\tau_2 T^2}-\frac{\pi R^2}{\tau_2\alpha'}(w'+\tau w)(w'+\bar{\tau}w)} Z^{tr}_{\epsilon_1,\epsilon_2}(\tau,\bar{\tau};m), \tag{5.1}$$

where[3]

$$m^2 = \mu^2 \Bigg(\frac{1}{(2\pi)^2 \tau_2^2 T^2} (k' - \tau k)(k' - \bar{\tau} k) - \frac{iR}{2\pi\tau_2^2 T} \left(2kw\tau\bar{\tau} - (\tau+\bar{\tau})(k'w + kw') + 2k'w'\right) - \\ - \frac{R^2}{\tau_2^2} (w' - \tau w)(w' - \bar{\tau} w) \Bigg) , \tag{5.2}$$

and

$$Z^{tr}_{a,b}(\tau, \bar{\tau}; m) = \frac{\Theta_{(a,b)}(\tau, \bar{\tau}; m)^4}{\Theta_{(0,0)}(\tau, \bar{\tau}; m)^4} , \tag{5.3}$$

as was defined in [24, 27]. This partition function corresponds to space-like compactification of the y coordinate near $r = 0$ so as to provide a natural normalization to the partition function. It is related to the light-cone compactified case of [27] by infinite boost accompanied by a scaling of R and μ. To take the flat space limit, one should replace the divergent factor

$$Z_0 = \frac{1}{(2\pi)^2 \tau_2^2 m^2} = \frac{1}{(2\pi)^2 \tau_2^2 m^2} , \tag{5.4}$$

which comes from the trace over the bosonic zero-mode oscillator

$$Z_T = Z_0 Z'_T , \tag{5.5}$$

by the standard zero-mode factor

$$Z_0 = \frac{V_2}{(2\pi)^2 \tau_2 \alpha'} . \tag{5.6}$$

The Hagedorn temperature can be computed by analyzing the large τ_2 behavior of the τ integral as before. This time, however, one finds that the Hagedorn temperature, given implicitly by the relation

$$\frac{1}{8\pi^2 \alpha' T_H^2} - 8 \left(\Delta \left(\frac{\mu}{2\pi T_H} ; \frac{1}{2} \right) - \Delta \left(\frac{\mu}{2\pi T_H} ; 0 \right) \right) = 0 , \tag{5.7}$$

does depend on the strength of the background RR flux μ, as was found earlier in similar models [25]–[28]. We have used the notation of [27] where

$$\Delta(m; a) = -\frac{1}{2\pi^2} \sum_{n=1}^{\infty} \int_0^{\infty} ds \, e^{-sn^2 - \frac{\pi^2 m^2}{s}} \cos(2\pi n a) . \tag{5.8}$$

For small μl_s, the Hagedorn temperature T_H is of the order of the string scale $1/l_s$. For large μl_s, T_H grows like

$$T_H(\mu) \sim \frac{\mu}{2\log(\mu l_s)} \left(1 + \mathcal{O}\left(\frac{1}{\log(\mu l_s)} \right) \right) . \tag{5.9}$$

To test the correspondence principle, first note that the entropy of the black hole

$$S = \frac{1}{G_9} (G_9 E)^{7/6} \tag{5.10}$$

[3]Note that we are using μ which differ in normalization with what was used in [27] by a factor of $\sqrt{2}$.

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and the entropy of the excited strings

$$S = \frac{E}{T_H(\mu)} \tag{5.11}$$

are in agreement precisely when

$$r_H = (G_9 E)^{1/6} = \frac{1}{T_H(\mu)} . \tag{5.12}$$

The correspondence principle requires that this is precisely the point where the curvature of the black-hole solution gets large in the unit set by the dynamics which gives rise to fluctuation in the background. In a sufficiently weakly coupled string theory, this fluctuation is associated with the stringy halo. The scale of this halo is set by the scale of the exponential growth in the density of stringy excitations. So if the correspondence principle is working, one expects the curvature near the horizon of the black hole to be of the order of magnitude set by $T_H(\mu)$ when the radius of the horizon is of the same order $r_H = 1/T_H(\mu)$.

By explicit calculation, one finds that the curvature of the black string solution in the Ramond-Ramond plane wave background scales like

$$R^2(r_H, \mu) = R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} = \begin{cases} \dfrac{1}{r_H^4} & \mu r_H \ll 1 \\ \dfrac{1}{\mu^2 r_H^6} & \mu r_H \gg 1 \end{cases} . \tag{5.13}$$

One then confirms that the dimensionless quantity

$$R^2(r_H, \mu) T_H(\mu)^4 , \tag{5.14}$$

at expected cross-over radius $r_H = 1/T_H(\mu)$, is always of order one regardless of the value of μl_s as long as one also treats $\log(\mu l_s)$ as a quantity of order one. It appears therefore that the correspondence principle is indeed working even though both the Hagedorn density and the curvature depend non-trivially on μ.

One additional condition, which was assumed implicitly in this discussion, requires that the string coupling be sufficiently weak, in order to ensure that the entropy of excited strings at the cross-over scale

$$\frac{E}{T_H} = \frac{r_H^6}{G_9 T_H} = \frac{1}{g^2 l_s^7 T_H^7} \gg 1 \tag{5.15}$$

is macroscopic.

6. Discussion

In this article, we computed the entropy as a function of energy for black strings in an asymptotically plane wave background and for string theory in the same plane-wave background. In fact, the thermal partition function can be computed for any of the "exactly solvable" backgrounds (4.1) in string theory considered in [16, 17], by substituting appropriate values to the general expression for the thermal partition function derived in section 4. For example, plane wave metric written in the form

$$ds^2 = -dt^2 + dy^2 + \sum_i (d\rho_i^2 + 2\eta\, \rho_i^2\, d\phi_i(dt + dy)) + \sum_{i=1}^{4} (d\rho_i^2 + \rho_i^2 d\phi_i^2) \tag{6.1}$$

which is useful for relating plane waves to Gödel universes, corresponds to setting

$$\alpha_i = 2\eta\,, \qquad \beta_i = 0\,, \qquad q_{+i} = 0\,, \qquad q_{-i} = 0\,. \tag{6.2}$$

Other related backgrounds, such as the Melvin universe, can also be considered. Indeed, a large class of black string solution in various asymptotic geometries can be constructed using a generalization of the Null Melvin Twist, and for each of these solutions, the corresponding thermal partition function for the free strings can be computed and compared.

A rigorous of mass and energy in a non-asymptotically flat space-time, however, can be rather subtle. In this paper, we used the notion of energy implied by the thermodynamic relation $dE = TdS$, and found that the correspondence principle is working well with this definition. This however should not be considered as an acceptable substitute for a careful definition of mass and energy in these spaces, and we hope that a more satisfying formulation of these quantities would appear in the literature in a due course (and hopefully in agreement with our expectations).

Acknowledgments

We would like to thank E. Gimon, V. Hubeny, O. Lunin, M. Rangamani, J. Russo, G. Shiu, Y. Sugawara, A. Tseytlin, and D. Vaman for discussions. The work of AH was supported in part by funds from the University of Wisconsin. The work of LAPZ was supported in part by the DOE grant DE-FG02-95ER40899. AH thanks the EFI at the University of Chicago and the MCTP at the University of Michigan, and LAPZ thanks the University of Wisconsin-Madison, for hospitality while part of this work was done.

A. Detailed computation of the thermal partition function

In this appendix, we describe the computation of thermal partition function. We follow much of the notations of [16] to which readers are referred for additional information.

Thermal partition function can be computed in the path integral formalism with only a minor modifications of the vacuum 1-loop partition function. Let us consider the case where only one set of α, β, q_+, and q_- are non-zero for the sake of illustration. One loop partition function was computed in [16] and is given by

$$\begin{aligned} Z_0 = {} & \frac{rV_0V_{22}}{(2\pi)^{23}\alpha'^{23/2}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{w,w'=-\infty}^{\infty} \left(4\tau_2^{-1} \int d\lambda d\bar{\lambda}\right) \left(\frac{e^{4\pi\tau_2}}{\tau_2^{12}} |f(e^{2\pi i\tau})|^{-48}\right) \times \\ & \times \frac{1}{(2\pi)^2} \exp[-\frac{\pi(\chi-\tilde{\chi})^2}{2\tau_2}] \frac{\tau_2|\theta_1'(0,\tau)|^2}{\theta_1(\chi|\tau)\theta_1(\tilde{\chi}|\bar{\tau})} \times \\ & \times \exp\left(-4\pi\tau_2^{-1}\left(\lambda\bar{\lambda} - \frac{1}{2}r(w'-\tau w)\bar{\lambda} + \frac{1}{2}r(w'-\bar{\tau}w)\lambda\right)\right). \end{aligned} \tag{A.1}$$

where χ is

$$\begin{aligned}\chi &= -\sqrt{\alpha'}[2\beta\lambda + q_+ r(w' - \tau w)] \\ \tilde{\chi} &= -\sqrt{\alpha'}[2\alpha\bar{\lambda} + q_+ r(w' - \bar{\tau} w)]\,,\end{aligned} \tag{A.2}$$

$r = R/\sqrt{\alpha'}$ is the radius of the compact y direction in string units, and V_0 is the infinite volume factor associated with the time direction. To compute the thermal partition function, evaluate the path integral around the background with boundary condition

$$t(\sigma_1 + n, \sigma_2 + m) = t(\sigma_1, \sigma_2) + \frac{1}{T} m \tag{A.3}$$

and integrate the modular parameter over the half-strip $\mathcal{E}$ instead of the fundamental domain $\mathcal{F}$. This makes the partition function take the form

$$\begin{aligned} Z_T = \; & \frac{r V_{22}}{(2\pi)^{23} \alpha'^{23/2} T} \int_{\mathcal{E}} \frac{d^2\tau}{\tau_2^2} \sum_{w,w'=-\infty}^{\infty} \left(4\tau_2^{-1} \int d\lambda d\bar{\lambda} \right) \left(\frac{e^{4\pi\tau_2}}{\tau_2^{12}} |f(e^{2\pi i \tau})|^{-48} \right) \times \\ & \times \frac{1}{(2\pi)^2} \exp\left[-\frac{\pi(\chi - \tilde{\chi})^2}{2\tau_2} \right] \frac{\tau_2 |\theta_1'(0,\tau)|^2}{\theta_1(\chi|\tau)\theta_1(\tilde{\chi}|\bar{\tau})} \times \\ & \exp\Bigg(-4\pi\tau_2^{-1} \Bigg(\lambda\bar{\lambda} - \left(\frac{1}{2} r(w' - \tau w) - \frac{i}{4\pi\sqrt{\alpha'}T} \right) \bar{\lambda} + \\ & \qquad + \left(\frac{1}{2} r(w' - \bar{\tau} w) + \frac{i}{4\pi\sqrt{\alpha'}T} \right) \lambda \Bigg) \Bigg)\,. \end{aligned} \tag{A.4}$$

where now

$$\begin{aligned}\chi &= -\sqrt{\alpha'}[2\beta\lambda + q_+ r(w' - \tau w)] + \frac{iq_-}{2\pi T} \\ \tilde{\chi} &= -\sqrt{\alpha'}[2\alpha\bar{\lambda} + q_+ r(w' - \bar{\tau} w)] + \frac{iq_-}{2\pi T}\,.\end{aligned} \tag{A.5}$$

Using the trick of [34], this partition function can be recast in a manifestly modular invariant form

$$\begin{aligned} Z_T = \; & \frac{r V_{22}}{(2\pi)^{23} \alpha'^{23/2} T} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{w,w'=-\infty}^{\infty} \left(4\tau_2^{-1} \int d\lambda d\bar{\lambda} \right) \left(\frac{e^{4\pi\tau_2}}{\tau_2^{12}} |f(e^{2\pi i \tau})|^{-48} \right) \times \\ & \times \frac{1}{(2\pi)^2} \exp\left[-\frac{\pi(\chi - \tilde{\chi})^2}{2\tau_2} \right] \frac{\tau_2 |\theta_1'(0,\tau)|^2}{\theta_1(\chi|\tau)\theta_1(\tilde{\chi}|\bar{\tau})} \times \\ & \times \exp\Bigg(-4\pi\tau_2^{-1} \Bigg(\lambda\bar{\lambda} - \left(\frac{1}{2} r(w' - \tau w) + \frac{i(k' - \tau k)}{4\pi\sqrt{\alpha'}T} \right) \bar{\lambda} + \\ & \qquad + \left(\frac{1}{2} r(w' - \bar{\tau} w) - \frac{i(k' - \bar{\tau} k)}{4\pi\sqrt{\alpha'}T} \right) \lambda \Bigg) \Bigg) \\ \chi = \; & -\sqrt{\alpha'}[2\beta\lambda + q_+ r(w' - \tau w)] + \frac{iq_-(k' - \tau k)}{2\pi T} \\ \tilde{\chi} = \; & -\sqrt{\alpha'}[2\alpha\bar{\lambda} + q_+ r(w' - \bar{\tau} w)] + \frac{iq_-(k' - \bar{\tau} k)}{2\pi T}\,. \end{aligned}$$

This expression is modular invariant. To see this, act with transformation

$$\tau \to -\frac{1}{\tau}, \qquad \lambda \to \frac{\lambda}{\tau}, \qquad \bar{\lambda} \to \frac{\bar{\lambda}}{\bar{\tau}}, \tag{A.6}$$

$$(k, k') \to (k', -k), \qquad (w, w') \to (w', -w). \tag{A.7}$$

which also causes χ and $\tilde{\chi}$ to transform according to

$$\chi \to \frac{\chi}{\tau}, \qquad \tilde{\chi} \to \frac{\tilde{\chi}}{\bar{\tau}}. \tag{A.8}$$

Of course, everything other than the τ is just an integration variable, so this implies that the integrand of the τ integral, when all of the other integrals are done, is a modular invariant function of τ. In the most general case where α_i, β_i, q_{+i}, and q_{-i} for all 12 planes are non-vanishing, one finds

$$\begin{aligned}
Z_T &= \frac{r}{(2\pi)\alpha'^{1/2}T}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\sum_{w,w'=-\infty}^{\infty}\left(4\tau_2^{-1}\int d\lambda d\bar{\lambda}\right)\left(\frac{e^{4\pi\tau_2}}{\tau_2^{12}}|f(e^{2\pi i\tau})|^{-48}\right)\times\\
&\quad\times\prod_{i=1}^{12}\left(\frac{1}{(2\pi)^2}\exp\left[-\frac{\pi(\chi_i-\tilde{\chi}_i)^2}{2\tau_2}\right]\frac{\tau_2|\theta_1'(0,\tau)|^2}{\theta_1(\chi_i|\tau)\theta_1(\tilde{\chi}_i|\bar{\tau})}\right)\times\\
&\quad\times\exp\Bigg(-4\pi\tau_2^{-1}\Bigg(\lambda\bar{\lambda}-\left(\frac{1}{2}r(w'-\tau w)+\frac{i(k'-\tau k)}{4\pi\sqrt{\alpha'}T}\right)\bar{\lambda}+\\
&\qquad\qquad+\left(\frac{1}{2}r(w'-\bar{\tau}w)-\frac{i(k'-\bar{\tau}k)}{4\pi\sqrt{\alpha'}T}\right)\lambda\Bigg)\Bigg)\\
\chi_i &= -\sqrt{\alpha'}[2\beta_i\lambda+q_{+i}r(w'-\tau w)]+\frac{iq_{-i}(k'-\tau k)}{2\pi T}\\
\tilde{\chi}_i &= -\sqrt{\alpha'}[2\alpha_i\bar{\lambda}+q_{+i}r(w'-\bar{\tau}w)]+\frac{iq_{-i}(k'-\bar{\tau}k)}{2\pi T}.
\end{aligned}\tag{A.9}$$

As a check, note that in the limit where all of the α_i, β_i, q_{+i}, and q_{-i} goes to zero, there will be a diverging factor of $1/(2\pi)^2(\chi_i\tilde{\chi}_i)$ for each of the 12 transverse planes coming from the contribution of the zero-mode to the path integral. Replacing this factor with the standard factor of $V_2/(2\pi)^2\alpha'\tau_2$, we recover the thermal partition function of bosonic strings in the conventional normalization.

Thermal partition function for similar exactly solvable type-II backgrounds can also be computed along these lines.

B. Oscillator computation of the thermal partition function

In this appendix, we describe the computation of the thermal partition function using the oscillator approach. The goal is to explicitly evaluate

$$Z_T = \frac{1}{T}\int_{\mathcal{E}}\frac{d^2\tau}{\tau_2}\sum_{m,w}\int\frac{d\varepsilon}{2\pi}e^{ik'\varepsilon/T}\,\mathrm{Tr}\left(e^{2\pi i(\tau L_0-\bar{\tau}\bar{L}_0)}\right)\Bigg|_{k'=-1} \tag{B.1}$$

for the background (4.1). Similar computation for the vacuum partition function was originally done by [16]. We therefore refer the reader to [16] for more information regarding conventions and notations. In actually evaluating the partition function, we first trace over the oscillators and then integrate over the zero modes. This is opposite of the order in which the computation was done in [16]. Doing the trace first actually clarifies certain technical aspect of this computation.

Let describe the case where only one set of α_i, β_i, q_{+i}, and q_{-i} is non-vanishing only for $i = 1$. One must then include the zero mode integral over p^a's

$$Z_T = \frac{V_{22}}{T} \int_{\mathcal{E}} \frac{d^2\tau}{\tau_2} \sum_{m,w} \int \frac{d\varepsilon}{2\pi} \frac{d^{22}p}{(2\pi)^{22}} e^{ik'\varepsilon/T}\, \mathrm{Tr}\left(e^{2\pi i(\tau L_0 - \bar{\tau}\bar{L}_0)} \right)\Bigg|_{k'=-1} . \tag{B.2}$$

The $\hat{L}_0$ and the $\hat{\bar{L}}_0$ operators are given by

$$\begin{aligned} \hat{L}_0 &= \frac{p_-^u p_-^v}{4\alpha'} + \frac{\alpha' p_a^2}{4} + N - \frac{1}{2}\gamma' \left(\hat{J}_R + \frac{1}{2} \right) - c_0 \\ \hat{\bar{L}}_0 &= \frac{p_+^u p_+^v}{4\alpha'} + \frac{\alpha' p_a^2}{4} + \bar{N} + \frac{1}{2}\gamma' \left(\hat{J}_L - \frac{1}{2} \right) - c_0 . \end{aligned} \tag{B.3}$$

where

$$\begin{aligned} c_0 &= 1 - \frac{1}{4}\gamma' + \frac{1}{8}\gamma'^2 \\ \hat{N} &= \sum_{n=1}^{\infty} n (b_{n+}^\dagger b_{n+} + b_{n-}^\dagger b_{n-} + a_{na}^\dagger a_{na}) \\ \hat{\bar{N}} &= \sum_{n=1}^{\infty} n (\tilde{b}_{n+}^\dagger \tilde{b}_{n+} + \tilde{b}_{n-}^\dagger \tilde{b}_{n-} + \tilde{a}_{na}^\dagger \tilde{a}_{na}) \\ \hat{J}_R &= -b_0^\dagger b_0 - \frac{1}{2} + \sum_{n=1}^{\infty} (b_{n+}^\dagger b_{n+} - b_{n-}^\dagger b_{n-}) \\ \hat{J}_L &= \tilde{b}_0^\dagger \tilde{b}_0 + \frac{1}{2} + \sum_{n=1}^{\infty} (\tilde{b}_{n+}^\dagger \tilde{b}_{n+} - \tilde{b}_{n-}^\dagger \tilde{b}_{n-}) \\ \frac{p_-^u p_-^v}{4\alpha'} + \frac{p_+^u p_+^v}{4\alpha'} &= \frac{\alpha'}{2} \Bigg(- \left(E - \frac{(a_- + c_-)\hat{J}}{2} \right)^2 + \left(p_y - \frac{(a_+ + c_+)\hat{J}}{2} \right)^2 + \\ &\qquad + \left(\frac{wR}{\alpha'} - \frac{(a_+ + c_+)\hat{J}}{2} \right)^2 - \frac{(a_- - c_-)^2 \hat{J}^2}{4} \Bigg) \\ \frac{p_-^u p_-^v}{4\alpha'} - \frac{p_+^u p_+^v}{4\alpha'} &= -mw + \frac{\gamma \hat{J}}{2} , \end{aligned} \tag{B.4}$$

and

$$\begin{aligned} \gamma &= (c_+ + a_+) wR + \frac{1}{2}(\alpha + \beta) s + \frac{1}{2}(\alpha - \beta) p \\ \gamma' &= 2\left(\frac{1}{2}\gamma - \left[\frac{1}{2}\gamma \right] \right) . \end{aligned} \tag{B.5}$$

The fact that the $\hat{L}_0$ operators depend on γ' which not a continuous function of γ may seem to suggest that the evaluation of the partition function cumbersome. One can however show that this expression is invariant under shift of γ' by 2, by computing the trace over oscillators first. To address this issue, let us introduce auxiliary variables $\mathcal{J}_L$ and $\mathcal{J}_R$ and write

$$\begin{aligned} Z_T &= \frac{V_{22}}{T}\int d\mathcal{J}_L\int d\mathcal{J}_R\int\frac{d^2\tau}{\tau_2}\int_{-\infty}^{\infty}\frac{d\varepsilon}{2\pi}\frac{d^{22}p}{(2\pi)^{22}}\sum_{m,w=-\infty}^{\infty} \\ &\quad e^{ik'\varepsilon/T}\,\mathrm{Tr}\,\delta(\hat{J}_L-\mathcal{J}_L)\delta(\hat{J}_R-\mathcal{J}_R)\exp[2\pi i(\tau L_0-\bar{\tau}\bar{L}_0)]\Big|_{k'=-1}, \end{aligned} \tag{B.6}$$

introduce an integral expression for the delta function

$$\delta(\hat{J}_R-\mathcal{J}_R)=\int d\chi\exp[-2\pi i\chi(J_R-\mathcal{J}_R)]\,,\qquad \delta(\hat{J}_L-\mathcal{J}_L)=\int d\tilde{\chi}\exp[-2\pi i\tilde{\chi}(J_L-\mathcal{J}_L)]\,, \tag{B.7}$$

and write γ in terms of $\mathcal{J}_{L,R}$ as

$$\gamma=(a_++c_+)wR+\alpha'[(c_+-a_+)p_y+(a_--c_-)E]+\frac{1}{2}\alpha'(a_+^2-a_-^2-c_+^2+c_-^2)(\mathcal{J}_L+\mathcal{J}_R)\,. \tag{B.8}$$

Using the trick of writing

$$\exp\left(\frac{\pi\tau_2}{2}\gamma^2\right)=\sqrt{\frac{\tau_2}{2}}\int d\nu\,\exp\left(-\frac{1}{2}\pi\tau^2\nu^2-\pi\tau^2\nu\gamma'\right) \tag{B.9}$$

introducing shift of variables

$$\mathcal{J}_{\mathcal{L}}=\mathcal{J}_L'-\frac{1}{2}\nu\,,\qquad \mathcal{J}_{\mathcal{R}}=\mathcal{J}_R'+\frac{1}{2}\nu\,, \tag{B.10}$$

and integrating out ν, one can show that

$$\begin{aligned} Z_T &= \frac{V_{22}}{T}\int\frac{d\tau^2}{\tau_2}\int d\chi\int d\tilde{\chi}\int d\mathcal{J}_L'\int d\mathcal{J}_R'\int_{-\infty}^{\infty}\frac{d\varepsilon}{2\pi}\frac{d^ap}{(2\pi)^a}\sum_{m,w=-\infty}^{\infty}e^{ik'\varepsilon/T}\times \\ &\quad\times\exp\left(-\frac{\pi(\chi-\tilde{\chi})^2}{2\tau_2}\right)\left(\mathrm{Tr}\exp\left[2\pi i(\tau(N-1)-\chi\hat{J}_R)\right]\right)\times \\ &\quad\times\left(\mathrm{Tr}\exp\left[-2\pi i(\bar{\tau}(\bar{N}-1)+\tilde{\chi}\hat{J}_L)\right]\right)\times \\ &\quad\times\exp\Bigg[2\pi i\chi\mathcal{J}_R'+2\pi i\tilde{\chi}\mathcal{J}_L'-\pi i\tau\gamma'\mathcal{J}_R'-\pi i\bar{\tau}\gamma'\mathcal{J}_L'+ \\ &\qquad+2\pi i\tau\left(\frac{p_-^up_-^v}{4\alpha'}+\frac{\alpha'p_a^2}{4}\right)-2\pi i\bar{\tau}\left(\frac{p_+^up_+^v}{4\alpha'}+\frac{\alpha'p_a^2}{4}\right)\Bigg]\Bigg|_{k'=-1}. \end{aligned} \tag{B.11}$$

The trace can be computed

$$\begin{aligned} &\mathrm{Tr}\left\{\exp\left[2\pi i(\tau(N-1)-\chi\hat{J}_R')-2\pi i(\bar{\tau}(\bar{N}-1)+\tilde{\chi}\hat{J}_L')\right]\right\}= \\ &\qquad=\frac{1}{(2\pi)^2}e^{4\pi\tau_2}|f(e^{2\pi i\tau})|^{-48}e^{i\pi(\chi-\tilde{\chi})\nu}\frac{|\theta_1'(0,\tau)|^2}{\theta_1(\chi,\tau)\theta_1(\tilde{\chi},\bar{\tau})}\,, \end{aligned} \tag{B.12}$$

giving

$$Z_T = \frac{V_{22}}{T}\int \frac{d\tau^2}{\tau_2}\int d\chi \int d\tilde{\chi}\int d\mathcal{J}'_L \int d\mathcal{J}'_R \int_{-\infty}^{\infty}\frac{d\varepsilon}{2\pi}\frac{d^a p}{(2\pi)^a}\sum_{m,w=-\infty}^{\infty} e^{ik'\varepsilon/t}\times$$
$$\times\left(\frac{1}{(2\pi)^2}e^{4\pi\tau_2}|f(e^{2\pi i\tau})|^{-48}\exp\left(-\frac{\pi(\chi-\tilde{\chi})^2}{2\tau_2}\right)\frac{|\theta'_1(0,\tau)|^2}{\theta_1(\chi,\tau)\theta_1(\tilde{\chi},\bar{\tau})}\right)\times$$
$$\times\exp\Bigg[2\pi i\chi\mathcal{J}'_R + 2\pi i\tilde{\chi}\mathcal{J}'_L - \pi i\tau\gamma'\mathcal{J}'_R - \pi i\bar{\tau}\gamma'\mathcal{J}'_L +$$
$$+ 2\pi i\tau\left(\frac{p^u_- p^v_-}{4\alpha'}+\frac{\alpha' p_a^2}{4}\right) - 2\pi i\bar{\tau}\left(\frac{p^u_+ p^v_+}{4\alpha'}+\frac{\alpha' p_a^2}{4}\right)\Bigg]\Bigg|_{k'=-1}. \qquad \text{(B.13)}$$

In this form, it can be readily verified that a shift $\gamma' \to \gamma' + 2$ can be canceled by a shift of integration variables

$$\chi \to \chi + \tau\,, \qquad \tilde{\chi} \to \bar{\chi} + \bar{\tau}\,. \qquad \text{(B.14)}$$

Now that we see that γ' can freely be replaced by γ, let us integrate out the zero modes ε and p_a, which is a gaussian integral, and Poisson resum m. To evaluate the integrals further, it is convenient to introduce an auxiliary variable by multiplying the integrand by

$$1 = 4\tau_2^{-1}\int d\lambda d\bar{\lambda}\times \qquad \text{(B.15)}$$
$$\times\exp\left(-4\pi\tau_2^{-1}[\lambda - \frac{1}{2}r(w'-\tau w) + i\tau_2\sqrt{\alpha'}\alpha\mathcal{J}'_L][\bar{\lambda}+\frac{1}{2}r(w'-\bar{\tau}w) - i\tau_2\sqrt{\alpha'}\beta\mathcal{J}'_R]\right).$$

Then, $\mathcal{J}'_L$ and $\mathcal{J}'_R$ integrals give rise to a delta function which constrain χ and $\tilde{\chi}$. As a result of these integrations, one finally arrives at

$$Z_T = \frac{rV_{22}}{(2\pi)^{23}\alpha'^{23/2}T}\int_{\mathcal{E}}\frac{d^2\tau}{\tau_2^2}\sum_{w,w'=-\infty}^{\infty}\left(4\tau_2^{-1}\int d\lambda d\bar{\lambda}\right)\left(\frac{e^{4\pi\tau_2}}{\tau_2^{12}}|f(e^{2\pi i\tau})|^{-48}\right)\times$$
$$\times\frac{1}{(2\pi)^2}\exp[-\frac{\pi(\chi-\tilde{\chi})^2}{2\tau_2}]\frac{\tau_2|\theta'_1(0,\tau)|^2}{\theta_1(\chi|\tau)\theta_1(\tilde{\chi}|\bar{\tau})}\times$$
$$\exp\left(-4\pi\tau_2^{-1}\left(\frac{k'^2}{16\pi^2\alpha' T^2}+\lambda\bar{\lambda}-\frac{1}{2}r(w'-\tau w)\bar{\lambda}+\frac{1}{2}r(w'-\bar{\tau}w)\lambda\right)\right)\Bigg|_{k'=-1} \qquad \text{(B.16)}$$

where

$$\chi = -\sqrt{\alpha'}[2\beta\lambda + q_+ r(w'-\tau w)] + \frac{ic_- k'}{2\pi T}$$
$$\tilde{\chi} = -\sqrt{\alpha'}[2\alpha\bar{\lambda} + q_+ r(w'-\bar{\tau}w)] + \frac{ia_- k'}{2\pi T}\,. \qquad \text{(B.17)}$$

Up to a shift in λ and $\bar{\lambda}$, this is identical to (A.4) and (A.5) that was presented in the previous section.

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